

Lovász Local Lemma (LLL)

• 'bad' events $A_1, \dots, A_n$

wts possible that none occur.

i.e. want $P(\bigcap_{i=1}^n A_i^c) > 0$.

as long as no bad event is certain.

• indep if $A_1, \dots, A_n$ indep.

then $P(\bigcap_{i=1}^n A_i^c) = \prod P(A_i^c) = \prod (1 - P(A_i)) > 0$.

• union if $\sum_{i=1}^n P(A_i) < 1$ then $P(\bigcap_{i=1}^n A_i^c) = 1 - P(\bigcup_{i=1}^n A_i) \leq \sum_{i=1}^n P(A_i) < 1$

union bound.

LLL tool for the case $A_1, \dots, A_n$ "nearly indep"

i.e. each A_i only depends on few other A_j 's, indep from rest.

Def

event A indep. from a family of events $(B_1, \dots, B_k)$

if $\forall S, T \subseteq [k]$ with S, T disjoint

$$P(A \mid (\bigcap_{i \in S} B_i) \cap (\bigcap_{j \in T} B_j^c)) = P(A).$$

(warning) not the same as pairwise-indep. !!

Example flip two coins (indep, prob $1/2/1/2$ for H/T).

A_{H_1} = first is heads $\{HH, HT\}$

A_{H_2} = second " " $\{HH, TH\}$

A_{same} = both coins same $\{HH, TT\}$.

then $A_{H_1} \perp A_{H_2}$, $A_{H_1} \perp A_{same}$, $A_{H_2} \perp A_{same}$

$$P(A_{H_1} \cap A_{same}) = P(\{HH\}) = 1/4.$$

$$P(A_{H_1}) \cdot P(A_{same}) = P(\{HH, HT\}) \cdot P(\{HH, TT\}) = 1/4$$

but A_{same} not indep from (family) A_{H_1}, A_{H_2}

$$P(A_{same} | A_{H_1} \cap A_{H_2}) = \text{(take } S = \{1, 2\})$$

Defⁿ

directed graph



$$D = ([n], E)$$

A digraph is called a dependency digraph

for events $A_1, \dots, A_n$ if for every i ;

$\rightarrow$ event A_i is indep of family $\{A_j : j \neq i, i \nrightarrow j\}$.

(i.e. $\{j_1, \dots, j_k\}$ non-ablts of i then
require that A_i indep of $\{A_{j_1}, \dots, A_{j_k}\}$.)

$\nearrow$
not nec. unique.

LLL (gen. form).Let D be a dep. digraph for events $A_1, \dots, A_n$.Supp. $x_1, \dots, x_n \in [0, 1)$ satisfy

$$P(A_i) \leq x_i \prod_{j: i \rightarrow j} (1 - x_j) \quad \forall i.$$

Then

$$P\left(\bigcap_{i=1}^n A_i^c\right) \geq \prod_{i=1}^n (1 - x_i) > 0$$

ProofClaim For any $S \subsetneq [n]$, any $i \notin S$,

$$P(A_i^c \mid \bigcap_{j \in S} A_j^c) \geq 1 - x_i.$$

 $S = \emptyset$,

$$\bigcap_{j \in \emptyset} A_j^c = \Omega$$

$$P(A_i^c \mid \Omega) = P(A_i^c) \geq 1 - x_i$$

Then claim $\Rightarrow$ thm by 'telescoping products'

$$\begin{aligned}
 P\left(\bigcap_{i=1}^n A_i^c\right) &= \overset{\geq 1-x_1}{P(A_1^c)} \cdot \overset{\geq 1-x_2}{P(A_2^c \mid A_1^c)} \cdot \overset{\geq 1-x_3}{P(A_3^c \mid A_1^c \cap A_2^c)} \cdot \dots \\
 &\quad \dots \cdot \overset{\geq 1-x_n}{P(A_n^c \mid A_1^c \cap \dots \cap A_{n-1}^c)} \\
 &\geq \prod_{i=1}^n (1 - x_i).
 \end{aligned}$$

Claim (repeat) for any $S \subseteq [n]$, any $i \notin S$,

$$P(A_i^c \mid \bigcap_{j \in S} A_j^c) \geq 1 - x_i.$$

Pf of claim $\rightarrow$ proceed by induction on S .

base

$$|S| = 0$$

$$P(A_i^c \mid \bigcap_{j \in \emptyset} A_j^c) \stackrel{\text{depth}}{=} P(A_i^c) = 1 - P(A_i)$$

$$\leq x_i \prod_{j: i \rightarrow j} (1 - x_j) \quad \forall i$$

dependency digraph.

inductive step

assume claim $\forall S: |S| \leq r-1$.

take

$|S| = r$, $i \notin S$. Let $S_1 = \{j \in S : i \rightarrow j\}$ "dep. part"
 $S_0 = \{j \in S : i \not\rightarrow j\}$ "indep part"

$$\text{w.t.s } P(A_i \mid \underbrace{\bigcap_{j \in S_1} A_j^c}_=: B \cap \underbrace{\bigcap_{j \in S_0} A_j^c}_=: C) \leq x_i$$

$$P(A_i \mid B \cap C) = \frac{P(A_i \cap B \mid C)}{P(B \mid C)} \leq \frac{P(A_i \mid C)}{P(B \mid C)} = \frac{P(A_i)}{P(B \mid C)} \quad (1)$$

$$P(A_i) \leq x_i \prod_{j: i \rightarrow j} (1 - x_j) \quad (2)$$

Note if $S_1 = \emptyset$ then $P(A_i \mid \bigcap_{j \in S_0} A_j^c) = P(A_i) \leq x_i$ $\therefore$ done, so may assume $|S_0| \leq r-1$.

Proof of claim (ct'd).

$$\begin{aligned}
 P(B|C) &= P(A_{S_1}^c \cap A_{S_2}^c \cap \dots \cap A_{S_n}^c | C) \\
 &= P(A_{S_1}^c | C) \cdot P(A_{S_2}^c | A_{S_1}^c \cap C) \cdot \dots \cdot P(A_{S_n}^c | A_{S_1}^c \cap \dots \cap A_{S_{n-1}}^c \cap C)
 \end{aligned}$$

$\uparrow$ $\leq r-1$ events $\leq r-1$ $\uparrow$ $\leq r-1$

by IH.

$$\geq (1-x_{S_1})(1-x_{S_2}) \dots (1-x_{S_n}) = \prod_{j: i \rightarrow j} (1-x_j).$$

Hence with (1), (2).

$$P(A_i | B \cap C) \stackrel{(1)}{\leq} \frac{P(A_i)}{P(B|C)} \leq \frac{x_i \prod_{j: i \rightarrow j} (1-x_j)}{\prod_{j: i \rightarrow j} (1-x_j)} = x_i. \quad \square$$

Rem. $P(A|B)$ not defined if $P(B)$

Fix : read $P(A|B) \geq t$ as shorthand for ..

$$P(A \cap B) \geq t \cdot P(B). \quad (\text{then ok}).$$

+ for defⁿ of dep. digraph read $P(A|B) = P(A)$

as $P(A \cap B) = P(A) \cdot P(B)$ ← ok for $P(B) = 0$.